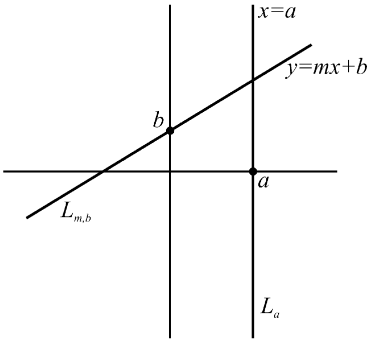
1 Definition and Models of Incidence Geometry

(1.1) Definition (geometry)
A geometry is a set $\mathcal{S}$ of points and a set $\mathcal{L}$ of lines together with relationships between the points and lines.

1. Find four points which belong to set $\mathcal{M} = \{(x,y) \in \mathbb{R}^2 \mid x = \sqrt{5}\}$.

2. Let $\mathcal{M} = \{(x,y) \mid x,y \in \mathbb{R}, x < 0\}$.
(i) Find three points which belong to set $\mathcal{N} = \{(x,y) \in \mathcal{M} \mid x = \sqrt{2}\}$.
(ii) Are points $P(3,3)$, $Q(6,4)$ and $R(-2,\frac{4}{3})$ belong to set $\mathcal{L} = \{(x,y) \in \mathcal{M} \mid y = \frac{1}{3}x + 2\}$?

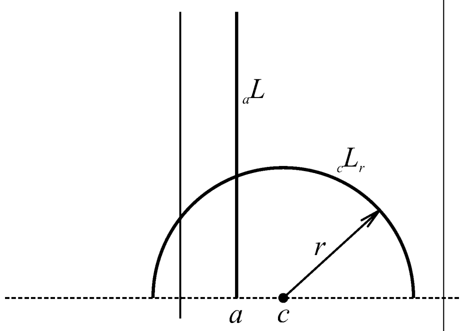


(1.2) Definition (Cartesian Plane)
Let $\mathcal{L}_E$ be the set of all vertical and non-vertical lines, L_a and $L_{k,n}$ where
vertical lines:
$L_a = \{(x,y) \in \mathbb{R}^2 \mid x = a,\ a$ is fixed real number$\}$,
non-vertical lines:
$L_{k,n} = \{(x,y) \in \mathbb{R}^2 \mid y = kx + n,\ k$ and n are fixed real numbers$\}$.
The model $\mathcal{C} = \{\mathbb{R}^2, \mathcal{L}_E\}$ is called the Cartesian Plane.

3. Let $\mathcal{C} = \{\mathbb{R}^2, \mathcal{L}_E\}$ denote Cartesian Plane.
(i) Find three different points which belong to Cartesian vertical line L_7.
(ii) Find three different points which belong to Cartesian non-vertical line $L_{15,\sqrt{2}}$.

4. Let P be some point in Cartesian Plane $\mathcal{C} = \{\mathbb{R}^2, \mathcal{L}_E\}$. Show that point P cannot lie simultaneously on both L_a and $L_{a'}$ (where $a \neq a'$).

(1.3) Definition (Poincaré Plane)
Let $\mathcal{L}_H$ be the set of all type I and type II lines, ${}_aL$ and ${}_pL_r$ where
type I lines:
${}_aL = \{(x,y) \in \mathbb{H} \mid x = a,\ a$ is a fixed real number$\}$,
type II lines:
${}_pL_r = \{(x,y) \in \mathbb{H} \mid (x-p)^2 + y^2 = r^2,\ p$ and r are fixed $\in \mathbb{R},\ r > 0\}$,
$\mathbb{H} = \{(x,y) \in \mathbb{R}^2 \mid y > 0\}$.
The model $\mathcal{H} = \{\mathbb{H}, \mathcal{L}_H\}$ will be called the Poincaré Plane.



5. Find the Poincaré line through
(i) points $P(1,2)$ and $Q(3,4)$;
(ii) points $M(1,2)$ and $N(3,4)$.

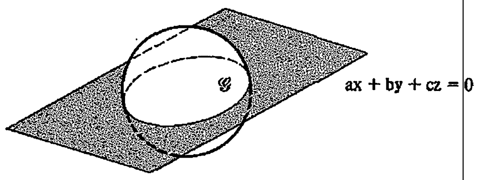
6. Let P and Q denote two different points in Cartesian Plane $\mathcal{C} = \{\mathbb{R}^2, \mathcal{L}_E\}$. Show that it is not possible for P and Q to lie simultaneously on two distinct lines L_a and $L_{k,n}$.

(1.4) Definition (unit sphere, plane)
The unit sphere in $\mathbb{R}^3$ is $S^2 = \{(x,y,z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\}$. A plane in $\mathbb{R}^3$ is a set of the form $\{(x,y,z) \in \mathbb{R}^3 \mid ax + by + cz = d$, where a,b,c,d are fixed real numbers, and not all of a,b,c are zero$\}$.
[GeoGebra: $x\hat{}2 + y\hat{}2 + z\hat{}2 = 1$, $x + y - (1/2) * z = 0$, find intersection]

(1.5) Definition (collinear set of points)
A set of points $\mathcal{P}$ is collinear if there is a line ℓ such that $\ell \subseteq \mathcal{P}$. $\mathcal{P}$ is non-collinear if $\mathcal{P}$ is not a collinear set.

(1.6) Definition (great circle)

A great circle, $\mathcal{G}$, of the sphere S^2 is the intersection of S^2 with a plane through the origin. Thus $\mathcal{G}$ is a great circle if there are $a, b, c \in \mathbb{R}$ not all zero, with



$$\mathcal{G} = \{(x, y, z) \in S^2 \mid ax + by + cz = 0\}.$$

7. Find a spherical line (great circle) through
(i) points $P(\frac{1}{2}, \frac{1}{2}, \sqrt{\frac{1}{2}})$ and $Q(1, 0, 0)$;
(ii) points $M(0, \frac{1}{2}, \frac{\sqrt{3}}{2})$ and $N(0, -1, 0)$.

8. Show by example that there are (at least) three non-collinear points in the Cartesian Plane.

9. Verify that $P(x_1, y_1)$ and $Q(x_2, y_2)$ do lie on ${}_pL_r$, where p and r are given by

$$p = \frac{y_2^2 - y_1^2 + x_2^2 - x_1^2}{2(x_2 - x_1)}, \qquad r = \sqrt{(x_1 - p)^2 + y_1^2}.$$

10. Let P and Q denote two different points in Cartesian Plane $\mathcal{C} = \{\mathbb{R}^2, \mathcal{L}_E\}$ which do not belong to the same vertical line. Show that P and Q cannot lie simultaneously on both $L_{k,n}$ and $L_{m,b}$.

11. Prove that if P and Q are distinct points in $\mathbb{H}$ then they cannot lie simultaneously on both ${}_aL$ and ${}_pL_r$.

12. Show by example that there are (at least) three non-collinear points in the Poincaré Plane.

(1.7) Definition (Riemann Sphere)

Let $\mathcal{L}_R$ be the set of great circles on S^2. The model $\mathcal{R} = \{S^2, \mathcal{L}_R\}$ is called the Riemann Sphere.

13. Explain, is it possible and are there two points on Riemann Sphere which lie simultaneously on two different spherical lines (great circles). If such two points exist, write them down, and find what are spherical lines which goes through that two points.

(1.8) Definition (abstract geometry)

An abstract geometry $\mathcal{A}$ consists of a set $\mathcal{S}$, whose elements are called points, together with a collection $\mathcal{L}$ of non-empty subsets of $\mathcal{S}$, called lines, such that:
(i) For every two points $A, B \in \mathcal{S}$ there is a line $\ell \in \mathcal{L}$ with $A \in \ell$ and $B \in \ell$.
(ii) Every line has at least two points.

14. Show that the Cartesian Plane $\mathcal{C} = \{\mathbb{R}^2, \mathcal{L}_E\}$ is an abstract geometry.

15. Show that the Poincaré Plane $\mathcal{H} = \{\mathbb{H}, \mathcal{L}_H\}$ is an abstract geometry.

16. Show that the Riemann Sphere $\mathcal{R} = \{S^2, \mathcal{L}_R\}$ is an abstract geometry.

17. Let P and Q be two points in $\mathbb{H}$ which don't lie on same vertical line. Use your knowledge from high school and find intersection of the Euclidean perpendicular bisector of the Euclidean line segment from P to Q with x-axis.

18. Let $P, Q \in \mathbb{H}$ denote two different points, and let $p(P, Q) = {}_cL_r$ (points P and Q lie on Poincaré line ${}_cL_r$). Use your knowledge of Euclidean geometry to prove that c is the x-coordinate of the intersection of the Euclidean perpendicular bisector of the Euclidean line segment from P to Q with x-axis.

(1.9) Definition (incidence geometry)

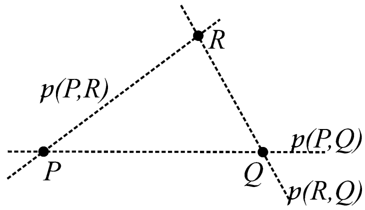
An abstract geometry $\{\mathcal{S}, \mathcal{L}\}$ is an incidence geometry if
(i) Every two distinct points in $\mathcal{S}$ lie on a unique line.
(ii) There exists a set of three non-collinear points.

Notation. If $\{\mathcal{S}, \mathcal{L}\}$ is an incidence geometry and $P, Q \in \mathcal{S}$ then the unique line p on which both P and Q lie will be written $p = p(P, Q)$.

19. Show that the Cartesian Plane $\mathcal{C} = \{\mathbb{R}^2, \mathcal{L}_E\}$ is an incidence geometry.

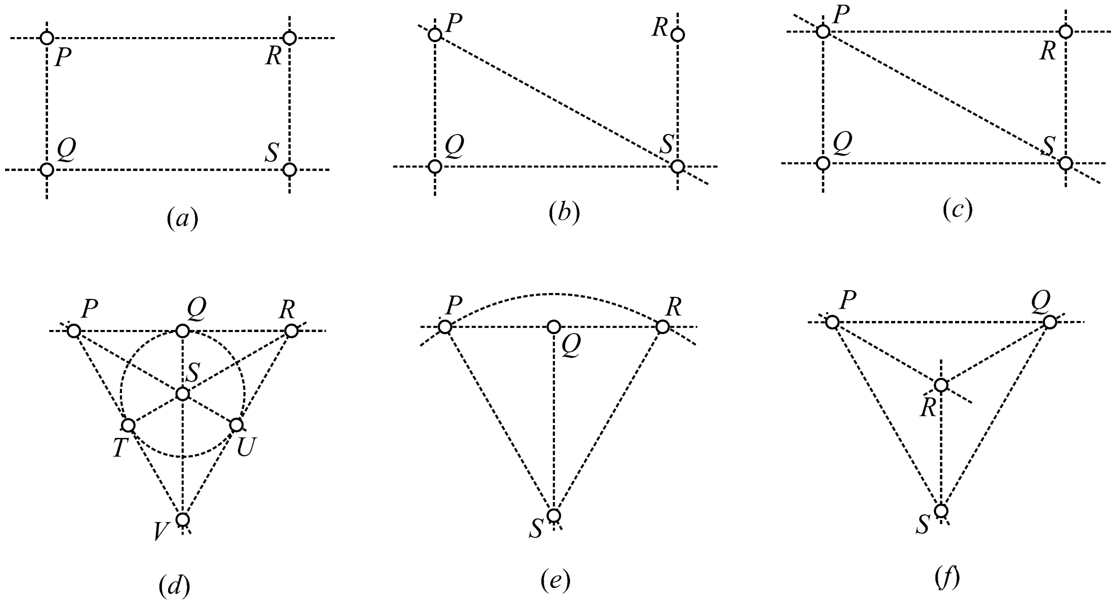
20. Show that the Poincaré Plane $\mathcal{H} = \{\mathbb{H}, \mathcal{L}_H\}$ is an incidence geometry.

21. Let $\mathcal{S} = \{P, Q, R\}$ and $\mathcal{L} = \{p(P,Q), p(P,R), p(Q,R)\}$. Show that $\{\mathcal{S}, \mathcal{L}\}$ is an incidence geometry. Note that this example has only finitely many (in fact, three) points. It may be pictured as in figure on right side. It is called the 3-point geometry. The dotted lines indicate which points lie on the same line.



22. Let $\mathcal{S} = \mathbb{R}^2 - \{(0,0)\}$ and $\mathcal{L}$ be the set of all Cartesian lines which lie in $\mathcal{S}$. Show that $\{\mathcal{S}, \mathcal{L}\}$ is not an incidence geometry.

23. Some finite geometries are defined pictorially (as in the 3-point geometry of Problem 21) by figure below.



(i) In each example list the set of lines.
(ii) Which of these geometries are abstract geometries?
(iii) Which of these geometries are incidence geometries?

”Prove” may mean ”find a counterexample”.

24. Let $\{\mathcal{S}_1, \mathcal{L}_1\}$ and $\{\mathcal{S}_2, \mathcal{L}_2\}$ be abstract geometries. Let $\mathcal{S} = \mathcal{S}_1 \cup \mathcal{S}_2$ and $\mathcal{L} = \mathcal{L}_1 \cup \mathcal{L}_2$. Prove that $\{\mathcal{S}, \mathcal{L}\}$ is an abstract geometry.

25. Let $\{\mathcal{S}_1, \mathcal{L}_1\}$ and $\{\mathcal{S}_2, \mathcal{L}_2\}$ be abstract geometries. Let $\mathcal{S} = \mathcal{S}_1 \cap \mathcal{S}_2$ and $\mathcal{L} = \mathcal{L}_1 \cap \mathcal{L}_2$. Prove that $\{\mathcal{S}, \mathcal{L}\}$ is an abstract geometry.

(1.10) Definition (parallel lines)
If ℓ_1 and ℓ_2 are lines in an abstract geometry then ℓ_1 is parallel to ℓ_2 (written $\ell_1 \parallel \ell_2$) if either $\ell_1 = \ell_2$ or $\ell_1 \cap \ell_2 = \emptyset$.

26. Let ℓ_1 and ℓ_2 be lines in an incidence geometry. Show that if $\ell_1 \cap \ell_2$ has two or more points then it $\ell_1 = \ell_2$.

27. Find all lines through $P(0,1)$ which are parallel to the vertical line L_6 in the Cartesian Plane $\mathcal{C} = \{\mathbb{R}^2, \mathcal{L}_E\}$.

28. Find all lines in the Poincaré Plane $\mathcal{H} = \{\mathbb{H}, \mathcal{L}_H\}$ through $P(0,1)$ which are parallel to the type I line ${}_6L$.

29. Find all lines of Riemann Sphere $\mathcal{R} = \{S^2, \mathcal{L}_R\}$ through $N(0,0,1)$ which are parallel to the spherical line (great circle) $\mathcal{G} = \{(x,y,z) \in S^2 \mid z = 0\}$.

(1.11) Definition (equivalence relation)

An equivalence relation on a set X is a relation $R \subseteq X \times X$ such that

(i) $(x,x) \in R$ for all $x \in X$ (reflexive property),

(ii) $(x,y) \in R$ implies $(y,x) \in R$ (symmetric property),

(iii) $(x,y) \in R$ and $(y,z) \in R$ imply $(x,z) \in R$ (transitive property).

Given an equivalence relation R on a set X, we usually write $x \sim y$ instead of $(x,y) \in R$. If the equivalence relation already has an associated notation such as $=$, $\equiv$, or $\cong$, we will use that notation.

30. Let $\{\mathcal{S},\mathcal{L}\}$ be an abstract geometry. If ℓ_1 and ℓ_2 are lines in $\mathcal{L}$ we write $\ell_1 \sim \ell_2$ if ℓ_1 is parallel to ℓ_2. Prove that $\sim$ is an equivalence relation. If $\{\mathcal{S},\mathcal{L}\}$ is the Cartesian Plane then each equivalence class can be characterized by a real number or infinity. What is this number?

31. There is a finite geometry with 7 points such that each line has exactly 3 points on it. Find this geometry. How many lines are there?

32. Let $\mathcal{S} = \mathbb{R}^2$ and, for a given choice of a, b, and c, let

$$J_{a,b,c} = \{(x,y) \in \mathbb{R}^2 \mid ax + by = c\}.$$

Let $\mathcal{L}_J$ be the set of all $J_{a,b,c}$ with at least one of a and b nonzero. Prove that $\{\mathbb{R}^2, \mathcal{L}_J\}$ is an incidence geometry. (Note that this incidence geometry gives the same family of lines as the Cartesian Plane. The point here is that there are different ways to describe the set of lines of this geometry.)

33. Define a relation $\sim$ on S^2 as follows. If $A = (x_1, y_1, z_1)$ and $B = (x_2, y_2, z_2)$ then $A \sim B$ if either $A = B$ or $A = -B = (-x_2, -y_2, -z_2)$. Prove $\sim$ is an equivalence relation.

34. Let $\{\mathcal{S},\mathcal{L}\}$ be an abstract geometry and assume that $\mathcal{S}_1 \subseteq \mathcal{S}$. We define an $\mathcal{S}_1$-line to be any subset of $\mathcal{S}_1$ of the form $\ell \cap \mathcal{S}_1$ where ℓ is a line of $\mathcal{S}$ and where $\ell \cap \mathcal{S}_1$ has at least two points. Let $\mathcal{L}_1$ be the collection of all $\mathcal{S}_1$-lines. Prove that $\{\mathcal{S}_1,\mathcal{L}_1\}$ is an abstract geometry. $\{\mathcal{S}_1,\mathcal{L}_1\}$ is called the geometry induced from $\{\mathcal{S},\mathcal{L}\}$.

Solutions: **1.** [for examle $P(\sqrt{5},0), Q(\sqrt{5},7), R(\sqrt{5},\sqrt{5}), M(\sqrt{5},\frac{-3}{2})$]. **2.** [$\mathcal{N} = \emptyset; P,Q \notin \mathcal{L}, R \in \mathcal{L}$] **3.** [for example $A(7,\frac{3}{2}), B(7,-4), C(7,0)$; $P(1,15+\sqrt{2}), Q(0,\sqrt{2}), R(-\sqrt{2}/15,0)$] **4.** [suppose the contrary, $P \in L_a, P \in L_{a'} \Rightarrow a = a'$, contradiction] **5.**(i) [${}_5L_{2\sqrt{5}}$; ${}_5L_{\sqrt{10}}$] (ii) [GeoGebra: $(x-5)\text{^}2 + y\text{^}2 = (2*sqrt(5))\text{^}2$, $x = 1$, $x = 3$...] **6.** [suppose the contrary] **7.**(i) [$\mathcal{G} = \{(x,y,z) \in \mathbb{R}^3 \,|\, x^2+y^2+z^2 = 1, -\sqrt{2}y + z = 0\}$] (ii) [$\mathcal{G} = \{(0,y,z) \in \mathbb{R}^3 \,|\, y^2+z^2 = 1\}$] [GeoGebra: $x\text{^}2+y\text{^}2+z\text{^}2 = 1$, (1/2,1/2,sqrt(1/2)), (1,0,0), -sqrt(2)*y+z=0] **8.** [for example $P(7,3), Q(7,-2) \in L_7$, $R(-1,0) \notin L_7$, $P,Q \notin$ non-vertical line] **9.** [$(x_1-p)^2 + y_1 = r^2$, $(x_2-p)^2 + y_2 = r^2$] **10.** [suppose the contrary, $P,Q \in L_{k,n} \Rightarrow k = \frac{y_2-y_1}{x_2-x_1}, n = y_1 - kx_1$...] **11.** [suppose the contrary, $P,Q \in {}_aL \Rightarrow x_1 = x_2 = a$; $P,Q \in {}_pL_r \Rightarrow y_1 = y_2$...] **12.** [for example $P(7,1), Q(7,-2) \in {}_7L$, $R(8,1) \notin {}_7L$, $P,Q \notin {}_pL_r$] **13.** [$N(0,0,1)$, $S(0,0,-1)$, $\{(x,y,z) \in S^2 \,|\, y = 0\}$, $\{(x,y,z) \in S^2 \,|\, x = 0\}$] **14.** [$1^\circ\, x_1 = x_2$; $2^\circ\, x_1 \neq x_2$] **15.** [$1^\circ\, x_1 = x_2$; $2^\circ\, x_1 \neq x_2$] **16.** [$ax_1 + by_1 + cz_1 = 0$...] **17.** [$\frac{x-x_1}{x_2-x_1} = \frac{y-y_1}{y_2-y_1}$, $k = \frac{y_2-y_1}{x_2-x_1}$, $s: y = k_s x + n$, $k_s = -\frac{1}{k}$, $s: y = -\frac{x_2-x_1}{y_2-y_1}x + \frac{y_2^2-y_1^2+x_2^2-x_1^2}{2(y_2-y_1)}$] **18.** [$P(x_1,y_1)$, $Q(x_2,y_2)$, S midpoint of $\overline{PQ}$, $R(c,0)$, we want to show that R belong to the Euclidean perpendicular bisector of $\overline{PQ}$, $\triangle PSR \cong \triangle QSR$] **19.** [$1^\circ\, P,Q \in L_a, P,Q \in L_{a'}$; $2^\circ\, P,Q \in L_a, P,Q \in L_{m,b}$; $3^\circ\, P,Q \in L_{m,b}, P,Q \in L_{n,c}$] **20.** [$1^\circ\, P,Q \in {}_aL, P,Q \in {}_{a'}L$; $2^\circ\, P,Q \in {}_aL, P,Q \in {}_cL_r$; $3^\circ\, P,Q \in {}_cL_r, P,Q \in {}_dL_s$] **21.** [we need to show that two properties from definition of incidence geometry are satisfied] **22.** [property (i) from definition of incidence geometry is not satisfied] **23.**(i) [(d) $p(P,Q,R)$, $p(P,T,V)$, $p(P,S,U)$...] (ii) [(d), (e), (f)] (iii) [(d), (f)] **24.** [$A \in \mathcal{S}_1$, $B \in \mathcal{S}_2$, $A,B \notin \mathcal{S}_1 \cap \mathcal{S}_2$] **25.** [look at figures (e) and (f) from Problem 23] **26.** [$P \in \ell_1 \wedge Q \in \ell_1 \Rightarrow \ell_1 = p(P,Q)$, $P \in \ell_2 \wedge Q \in \ell_2 \Rightarrow \ell_2 = p(P,Q)$] **27.** [$L_0$, $L_6 \cap L_{k,1} = \emptyset$] **28.** [${}_aL \Rightarrow a = 0$, ${}_cL_r \Rightarrow c^2 + 1^2 = r^2$, $c < 35/12$; GeoGebra: $r = \text{sqrt}(c\text{^}2+1)$, $y = \text{sqrt}(r\text{^}2 - (x-c)\text{^}2)$] **29.** [$c = 0$, $ax + by = 0$, such line does not exist] **30.** [$\ell_1 \cap \ell_2 = \emptyset$, $\ell_2 \cap \ell_3 = \emptyset$, $p(P,R) \parallel p(S,Q)$, $p(P,R) \parallel p(T,Q)$, $\sim$ is not an equivalence relation] **31.** [Fano plane] **32.** [$1^\circ x_1 = x_2$, $2^\circ y_1 = y_2$, $3^\circ x_1 \neq x_2$ and $y_1 \neq y_2$]